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On Finite Difference Fitted Schemes for Singularly Perturbed Boundary Value Problems with a Parabolic Boundary Layer

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A method to construct grid approximations for singularly perturbed boundary value problems for elliptic and parabolic equations, whose solutions contain a parabolic boundary layer, is considered. The grid approximations are based on the fitted operator method. Finite difference schemes, finite element, or finite volume techniques are included in the term grid approximation methods. It is shown that there exists no grid approximation method on uniform grids from the class of fitted operator methods, whose solutions converge, in the discrete maximum norm, uniformly with respect to the perturbation parameter to the solution of the boundary value problem.

1. INTRODUCTION

Effective numerical methods to approximate sufficiently smooth solutions of boundary values problems have been developed (see, for example, [7, 10]). Reducing the smoothness of the solution brings a decrease in accuracy, and even a complete loss of accuracy, of the approximate solution. Differential equations with a small parameter (the perturbation parameter ε) multiplying the highest derivatives are called singularly perturbed differential equations. The smoothness of the solutions of such singularly perturbed differential equations deteriorates when the parameter tends to zero (see, for example, [4, 6]). Therefore there is a need to develop, for such type of problems, special numerical methods, whose accuracy does not depend on the parameter value ε, i.e., methods that are convergence ε-uniformly.

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For the solution of various model singularly perturbed boundary value problems, in [2, 3, 6] special finite difference schemes are constructed and proved to be convergent ε-uniformly. These first strong results for problems with boundary layers belong to two different approaches, both widely used for the construction of special numerical methods. In [6] the coefficients of finite difference schemes are determinated (or are fitted) in such a way as to ensure, on an arbitrary grid (for example, on a uniform grid) ε-uniform convergency of the approximate solution. In this paper this class of methods is called fitted operator methods. In [3] using traditional finite difference operators, ε-uniform convergency of the approximate solution is achieved by the redistribution of the given mesh points. In this paper these methods shall be called fitted mesh methods.

Fitted operator methods are attractive and have been sufficiently widely developed. A bibliography on this subject and some results can be seen, for example, in [4, 9] and also [1, 5, 6]. These methods give an opportunity to obtain a solution on very simple meshes (e.g., uniform meshes).

In [13, 14] it was shown that the fitted operator method, applied to singularly perturbed boundary value problems, has a restricted area of application. For singularly perturbed parabolic equations, whose solutions have parabolic boundary layers (that is, a layer described by a parabolic equation), there does not exist a scheme of fitted operator type whose solutions converge to the solution of the boundary value problem ε-uniformly. From here it follows that for such problems to construct special finite difference schemes convergent ε-uniformly, the use of meshes condensing in the boundary layer is necessary. In [13, 14] this result was proved for finite difference schemes, for which the maximum principle was assumed.

In this paper the result is extended to finite difference schemes, for which the maximum principle is not assumed. Boundary value problems for elliptic and parabolic equations, whose solutions have a parabolic boundary layer, are studied. It is shown that under sufficiently natural assumptions there do not exist fitted operator schemes convergent ε-uniformly in the discrete maximum norm. This means that finite element methods are not suitable for singularly perturbed problems whose solution has a parabolic boundary layer, unless the mesh itself is fitted. So if one is interested in the nature of the solution in the neighbourhood of the boundary layer then norms other than the discrete maximum norm are not suitable for singularly perturbed problems (see [8, Chap. 3] for a discussion of these issues). The proof of statements from [13, 14] is given also; in [13, 14] only an outline of the proof was given.

In Section 2, the parabolic problem to be considered is stated and the concept of a fitted operator method is described. Also in this section, a theorem about the non-existence of a finite difference scheme, which satisfies a discrete maximum principle, that is uniformly convergent for a singularly perturbed parabolic problem with a parabolic boundary layer is stated. In Section 3, the proof of this theorem is given. In Section 4, the same result is proved for a finite difference scheme, not necessarily satisfying a discrete maximum principle. In Section 5, the corresponding result for a singularly perturbed elliptic equation with a parabolic layer is given.

For problems with a parabolic boundary layer, in particular, equations of parabolic type in [11, 12, 14], ε-uniformly convergent schemes using special condensing grids are constructed.

2. FITTED OPERATOR METHODS SATISFYING A DISCRETE MAXIMUM PRINCIPLE

On the half-strip $G = \{(x,t): 0 < x < \infty, 0 < t \le T\}$ with the boundary $S = S(G) = \overline{G} \setminus G$ we consider the Dirichlet problem for the singularly perturbed heat equation

$$Lu(x,t) \equiv \left\{ \varepsilon^2 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right\} u(x,t) = f(x,t), \qquad (x,t) \in G,\ \varepsilon \in (0,1]$$

$$u(x,t) = \varphi(x,t), \qquad (x,t) \in S. \tag{2.1}$$

Here $f(x,t)$, $(x,t) \in \overline{G}$, $\varphi(x,t), (x,t) \in S$ are sufficiently smooth functions.

In the neighbourhood of the lateral boundary $S^L = \{(x,t): x = 0, (x,t) \in S\}$ of the set G a parabolic boundary layer appears as $\varepsilon \to 0$, that is, a layer described by the homogeneous heat equation (2.6a).

We are interested in grid approximations of the boundary value problems (2.1) on rectangular grids. let

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0 \tag{2.2}$$

be a rectangular grid on $\overline{G}$, where $\overline{\omega}$ and $\overline{\omega}_0$ are, generally speaking, nonuniform grids on the half-axis $[0, \infty)$ and the segment $[0, T]$, respectively. By $N + 1$ and $N_0 + 1$ we denote respectively the minimal number of nodes in the grid $\overline{\omega}$ on the segment with unit length and the number of nodes in the grid $\overline{\omega}_0$. By h, τ we denote the maximal step-size of $\overline{\omega}$ and $\overline{\omega}_0$, $h \le MN^{-1}$, $\tau \le MN_0^{-1}$. Let $z(x,t), (x,t) \in \overline{G}_h$ be the solution of some finite difference scheme on the grid set $\overline{G}_h$. We say that the solution of this finite difference scheme converges ε-uniformly (in the discrete maximum norm) if for the function $z(x,t)$ the following estimate is fulfilled

$$\max_{\overline{G}_h} |u(x,t) - z(x,t)| \leq \lambda\big(N^{-1}, N_0^{-1}\big),$$

where as $N, N_0 \to \infty$ we have $\lambda(N^{-1}, N_0^{-1}) \to 0$ uniformly with respect to the parameter ε.

For problem (2.1) we wish to construct a special finite difference scheme convergent ε-uniformly in the discrete maximum norm. In the construction of a scheme we use the fitted operator method.

Let us describe a class of finite difference schemes (class A), in which we shall construct the fitted operator scheme for such a particular problem

$$Lu(x,t) = 0, \qquad (x,t) \in G, \tag{2.3a}$$

$$u(x,t) = \varphi_0(t), \qquad (x,t) \in S, \tag{2.3b}$$

where $\varphi_0(t) = \varphi(0,t)$, with $\varphi_0(0) = 0$. The solution of this problem is the function $V(x,t)$ which is a boundary layer type function. Note, that the singular part of the solution of problem (2.1) is the solution of the homogeneous equation (2.3a). The function $V(x,t)$ is also the principal term of the asymptotic expansion of the solution of problem (2.1), when $f(x,t) = 0$, $(x,t) \in S^L$.

On the set $\overline{G}$ we introduce the uniform rectangular grid

$$\overline{G}_h^u \tag{2.4}$$

with step-size h with respect to the space variable and step-size τ with respect to the time variable.

In the case of the grid $\overline{G}_h^u$ the solution $z(x,t)$ of a finite difference scheme converges ε-uniformly, if the following estimate is satisfied

$$\max_{\overline{G}_h} |u(x,t) - z(x,t)| \leq \lambda(h, \tau),$$

where $\lambda(h,\tau) \to 0$ uniformly with respect to the parameter ε as $h, \tau \to 0$.

We construct the finite difference scheme on a four-point stencil of implicit finite difference schemes. The finite difference equations are written out in the general form [10],

$$\Lambda z(x,t) \equiv \big\{A\delta_{x\bar{x}} + B\delta_{\bar{x}} - C - \delta_{\bar{t}}\big\} z(x,t) = D, \qquad (x,t) \in G_h, \tag{2.5}$$

where $\delta_{x\bar{x}} z(x,t) = z_{x\bar{x}}(x,t)$, $\delta_{\bar{x}} z(x,t) = z_{\bar{x}}(x,t)$, and $\delta_{\bar{t}} z(x,t) = z_{\bar{t}}(x,t)$ are the second order central and backward first difference derivatives, the coefficients A, B, C, D are functionals of the coefficients of Eq. (2.3a) and depend on $x, t, h, \tau, \varepsilon$; $G_h = G \cap \overline{G}_h^u$.

Instead of the variables x, t we shall use the stretched variables ξ, t; $\xi = \xi(x, \varepsilon) = \varepsilon^{-1}x$. The singular perturbed equation (2.3a) transforms to the regular heat equation

$$L^0 u^0(\xi, t) \equiv \left\{ \frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial t} \right\} u^0(\xi, t) = 0, \qquad (\xi, t) \in G_\xi; \quad (2.6a)$$

the sets G_ξ^0 in the variables ξ, t correspond to the sets $G^0 \subseteq \overline{G}$. We use the notation $v(x(\xi), t) = v^0(\xi, t)$. On the boundary S_ξ the function $u^0(\xi, t)$ takes the values

$$u^0(\xi, t) = \varphi_0(t), \qquad (\xi, t) \in S_\xi. \quad (2.6b)$$

Such a relation $u(x, t) = u^0(\varepsilon^{-1}x, t)$ is true, where $u(x, t)$ is the solution of problem (2.3).

On the grid $G_{h\xi}$, Eqs. (2.5) take the form

$$\Lambda z^0(\xi, t) \equiv \left\{ A_0^0 \delta_{\xi\bar{\xi}} + B_0^0 \delta_{\bar{\xi}} - C^0 - \delta_{\bar{t}} \right\} z^0(\xi, t) = D^0, \qquad (\xi, t) \in G_{h\xi}, \quad (2.7)$$

where $z^0(\xi, t) = z(\varepsilon\xi, t)$, $A_0^0(\xi, t, h_\xi, \tau, \varepsilon) = \varepsilon^{-2}A(\varepsilon\xi, t, \varepsilon h_\xi, \tau, \varepsilon)$, $B_0^0(\xi, t, h_\xi, \tau, \varepsilon) = \varepsilon^{-1}B(\varepsilon\xi, t, \varepsilon h_\xi, \tau, \varepsilon)$, $C^0(\xi, t, h_\xi, \tau, \varepsilon) = C(\varepsilon\xi, t, \varepsilon h_\xi, \tau, \varepsilon)$, $D^0(\xi, t, h_\xi, \tau, \varepsilon) = D(\varepsilon\xi, t, \varepsilon h_\xi, \tau, \varepsilon)$, $h_\xi = \varepsilon^{-1}h$.

Note, the parameter ε does not enter into the formulation of the problem (2.6). The parameter ε does not influence the grid sets $G_{h\xi}$ and $S_{h\xi}$; these sets are defined by the step-size h_ξ, τ only. Because problem (2.6), and consequently its solution, and also the grid $\overline{G}_{h\xi}$ are independent of the parameter ε, therefore it is natural to look for the coefficients of the finite difference equation (2.7), corresponding to differential equation (2.6a), in the form independent of the parameter ε

$$\begin{aligned} \Lambda^0 z^0(\xi, t) \equiv \Big\{ A_0^0(\xi, t, h_\xi, \tau) \delta_{\xi\bar{\xi}} + B_0^0(\xi, t, h_\xi, \tau)\, \delta_{\bar{\xi}} \\ - C^0(\xi, t, h_\xi, \tau) - \delta_{\bar{t}} \Big\} z^0(\xi, t) \\ = D^0(\xi, t, h_\xi, \tau), \qquad (\xi, t) \in G_{h\xi}. \quad (2.8) \end{aligned}$$

Equations (2.5) on the grid G_h^u in the variables x, t and Eqs. (2.8) on the grid $G_{h\xi}$ in the variables ξ, t are equivalent.

The condition of pointwise approximation on a smooth function of the operator L^0 by the operator Λ^0 [10] (or, that is, the condition of consistency for the coefficients of Eqs. (2.6a) and (2.8) [4]) brings us to the relations

$$\left|A_0^0(\xi,t,h_\xi,\tau)-1\right|,\left|B_0^0(\xi,t,h_\xi,\tau)\right|,\left|C^0(\xi,t,h_\xi,\tau)\right|,\left|D^0(\xi,t,h_\xi,\tau)\right|$$
$$\le \mu(h_\xi,\tau,\xi,t), \tag{2.9a}$$

where $\mu(h_\xi,\tau,\xi,t)\to 0$ as h_ξ, $\tau\to 0$ at a point $(\xi,t)\in G_{h\xi}$. The value $\mu(h_\xi,\tau,\xi,t)$ does not depend on the parameter ε.

We shall say that the operator Λ^0 approximates the operator L^0 on the set $G_\xi^*\subset\overline{G}_\xi$, if $\mu(h_\xi,\tau,\xi,t)$ does not depend on ξ,t for $(\xi,t)\in G_\xi^*$, that is,

$$\mu(h_\xi,\tau,\xi,t)=\lambda(h_\xi,\tau)\qquad\text{for } (\xi,t)\in G_\xi^*\cap G_{h\xi}. \tag{2.9b}$$

We shall assume that condition (2.9) is satisfied, where G_ξ^* belongs to an m-neighbourhood of the set S_ξ^L. In this class A we seek to construct fitted operator schemes.

The following theorem is valid [13, 14].

THEOREM 1. *In the class A (satisfy (2.9)) of finite difference schemes there does not exist a scheme, whose solution converges as $h,\tau\to 0$ to the solution of boundary problem (2.3) ε-uniformly.*

The proof of this theorem is given in Section 3.

Notation. Here and below by M, M_i (or $m, m_i m^i$) we denote sufficiently large (small) positive constants, which do not depend on the parameter ε and the mesh parameters.

Remark 2.1. The statement of Theorem 1 remains valid if the coefficients A_0^0, B_0^0, C^0, D^0 depend on the parameter ε.

Remark 2.2. Suppose fitted operator schemes are constructed on stencils of explicit four-point schemes (or implicit weighted six-point schemes [10]), and also the coefficients of these schemes in the variables ξ,t approximate the coefficients of Eq. (2.6a) on some subset from an m-neighbourhood of the side S_ξ^L uniformly. In this class there does not exist an ε-uniformly convergent finite difference scheme.

Remark 2.3. The results of Theorem 1 and the subsequent remarks can be explained as follows. All the solution of problem (2.3) (defined by various functions $\varphi_0(t)$) are singularly perturbed solutions. These solutions cannot be represented as a linear combination of a finite sum of some fixed basis functions of a boundary layer type. Here the number of basis functions is uncountable. For example, each function $\varphi_0(t)=t^n$, where n is a positive integer, generates its own basis function. Therefore by using a finite number of fitted coefficients, it is impossible to satisfy the difference equations (2.5) for all basis functions. This remark is related to the ideas of necessary conditions for uniform convergence (see, e.g., [4, 14]).

Remark 2.4. The conditions (2.9) imply fulfillment of the maximum principle for the finite difference scheme on the subset $G_h^* \equiv G^* \cap G_h \subset G_h$, $\{G^*\}_\xi = G_\xi^*$, for sufficiently small step-size h, τ of the grid $\overline{G}_h^u$. Indeed, let the condition (2.9) be satisfied. Then on the set $G_{h\xi}^*$, where $G_{h\xi}^* = G_\xi^* \cap G_{h\xi}$, the function $C^0(\xi, t, h_\xi, \tau)$ is bounded uniformly with respect to the parameter. We shall suppose

$$C^0(\xi, t, h_\xi, \tau) \geq 0, \qquad (\xi, t) \in G_{h\xi}^*. \tag{2.10}$$

If it is not satisfied, we pass from the functions $u(x,t)$ and $z(x,t)$ to the functions $u_\alpha(x,t) = u(x,t)\exp(-\alpha t)$ and $z_\alpha(x,t) = z(x,t)\exp(-\alpha t)$. For the function $z_\alpha^0(\xi, t) = z_\alpha(\varepsilon\xi, t)$ the equation is fulfilled

$$\left\{A_0^0 \delta_{\xi\bar{\xi}} + B_0^0 \delta_{\bar{\xi}} - \tilde{C}^0 - \tilde{P}^0 \delta_{\bar{t}}\right\} z_\alpha^0(\xi, t) = \tilde{D}^0, \qquad (\xi, t) \in G_{h\xi}. \tag{2.11}$$

Here

$$\tilde{C}^0 = C^0 + \exp(-\alpha t)\,\delta_{\bar{t}} \exp(\alpha t),$$

$$\tilde{P}_0 = \exp(-\alpha\tau), \qquad \tilde{D}^0 = D^0 \exp(-\alpha t).$$

The value α is chosen sufficiently large to satisfy the inequality

$$\tilde{C}^0 = \tilde{C}^0(\xi, t, h_\xi, \tau) \geq 0, \qquad (\xi, t) \in G_{h\xi}^*.$$

In the case of condition (2.10) such a variant of the maximum principle is valid. Let the set G_ξ^1 be a subset of the set $\overline{G}_\xi^*, \overline{G}_\xi^1 \subseteq \overline{G}_\xi^*$, $G_\xi^1 = D_\xi^1 \times (0, T]$, and let S_ξ^1 be its boundary $S_\xi^1 = \overline{G}_\xi^1 \setminus G_\xi^1$. Suppose that the boundary S^1 passes through the nodes of the grid $\overline{G}_h$, $\{S^1\}_\xi = S_\xi^1$. Denote $G_h^1 = G^1 \cap \overline{G}_h$, $S_h^1 = S^1 \cap \overline{G}_h$.

LEMMA 1. *Assume for the grid function* $w(\xi, t), (\xi, t) \in \overline{G}_\xi^1$ *the following relations are fulfilled*

$$\Lambda^0 w(\xi, t) \leq 0, (\xi, t) \in G_{h\xi}^1, \qquad w(\xi, t) \geq 0, (\xi, t) \in S_{h\xi}^1.$$

Then $w(\xi, t) \geq 0$, $(\xi, t) \in \overline{G}_{h\xi}^1$ *for* $h_\xi \leq m^1$, $\tau \leq m^2$, *where* m^1, m^2 *are sufficiently small numbers*.

Remark 2.5. We shall illustrate the validity of conditions (2.9) for the ordinary differential equation

$$\left\{\varepsilon^2 \frac{\partial^2}{\partial x^2} - c(x)\right\} u(x) = f(x), \qquad x \in D, \tag{2.12}$$

for which, as it is known [4, 1], in the case of the Dirichlet problem a fitted operator scheme exists. Here $D = (0, 1)$; $c(x) \geq c_0 > 0$, $x \in \overline{D}$. The grid equations of the fitted operator method have the form [4, 15]

$$\left\{\varepsilon^2\gamma(x)\,\delta_{x\bar{x}} - c(x)\right\}z(x) = f(x), \qquad x \in D_h. \tag{2.13}$$

Here $D_h = D \cap \overline{D}_h$, $\overline{D}_h$ is a uniform grid on $\overline{D}$ with step-size h; $\delta_{x\bar{x}}z(x)$ is the second order difference derivative, and the fitted coefficient $\gamma(x) = \gamma(x, h, \varepsilon)$ is defined by the formula

$$\gamma(x) = \frac{c(x)h^2}{4\varepsilon^2}\sinh^{-2}\frac{c^{1/2}(x)h}{2\varepsilon}, \qquad x \in D_h.$$

Let us introduce $\xi = \varepsilon^{-1}x$. Equations (2.12), (2.13) transform to

$$\left\{\frac{\partial^2}{\partial\xi^2} - c^0(\xi)\right\}u^0(\xi) = f^0(\xi), \qquad \xi \in D_\xi, \tag{2.14}$$

$$\left\{\gamma^0(\xi)\,\delta_{\xi\bar{\xi}} - c^0(\xi)\right\}z^0(\xi) = f^0(\xi), \qquad \xi \in D_{h\xi}, \tag{2.15}$$

where

$$\gamma^0(\xi) = \gamma^0(\xi, h_\xi) = \frac{c^0(\xi)h_\xi^2}{4}\sinh^{-2}\frac{\left(c^0(\xi)\right)^{1/2}h_\xi}{2}, \qquad \xi \in D_{h\xi}.$$

Note, the difference equation (2.14) does not depend on the parameter ε. The parameter ε does not influence also the coefficients and the right-hand side of Eq. (2.15). They are defined only by the functions $c^0(\xi)$, $f^0(\xi)$, and the step-size h_ξ.

For the function $\gamma^0(\xi) = \gamma^0(\xi, h_\xi)$ the following estimate is valid

$$\left|\gamma^0(\xi) - 1\right| \leq Mh_\xi^2, \qquad \xi \in D_{h\xi},$$

that is, the difference equations (2.13), (2.15) satisfy the analogue of condition (2.9), where $D_\xi^* = \overline{D}_\xi$.

3. THE PROOF OF THEOREM 1

The proof of the theorem is performed by the contradiction method. Assume that on the grid $\overline{G}_h^u$ there exists a finite difference scheme convergent ε-uniformly. According to condition (2.9) it follows that as h, $\tau \to 0$ the coefficients $C(x, t, h, \tau, \varepsilon)$, $D(x, t, h, \tau, \varepsilon) \to 0$ tend to zero uniformly with respect to the parameter on the set $G_h^1 \subseteq G^*$. We write out

Eq. (2.5) in the form

$$\Lambda_1 z(x,t) \equiv \left\{A\delta_{x\bar{x}} + B\delta_{\tilde{x}} - \delta_{\bar{t}}\right\}z(x,t) = \psi(x,t), \qquad (x,t) \in G_h^1, \tag{3.1}$$

where $\psi(x,t) = \psi(x,t,h,\tau,\varepsilon) = Cz(x,t) - D$, $|\psi(x,t,h,\tau,\varepsilon)| \le \lambda(h,\tau)$, $(x,t) \in G_h^1$. The function $z(x,t)$, that is, the solution of the grid problem with difference equations (3.1), is represented as a sum of functions $z(x,t) = z_1(x,t) + z_2(x,t)$, where $z_1(x,t)$ is the solution of the problem

$$\Lambda_1 z_1(x,t) = \psi(x,t), \qquad (x,t) \in G_h^1,\ z_1(x,t) = 0,\ (x,t) \in S_h^1.$$

Because $|z_1(x,t)| \le \lambda(h,\tau)$, $(x,t) \in \overline{G}_h^1$ (according to Lemma 1), if the scheme with difference equations (2.5) (Eq. (3.1)), convergent ε-uniformly on $\overline{G}_h^1$ exists, then the scheme with equations

$$\Lambda_1 z(x,t) = 0, \qquad (x,t) \in G_h^1,$$

convergent ε-uniformly on $\overline{G}_h^1$ also exists.

Consider the case when $B \equiv 0$, that is, the functions $z(x,t)$ and $z^0(\xi,t)$ are the solutions of the equations

$$\begin{aligned} \Lambda_2 z(x,t) &\equiv \left\{\varepsilon^2\gamma(x,t,h,\tau,\varepsilon)\delta_{x\bar{x}} - \delta_{\bar{t}}\right\}z(x,t) = 0, \qquad (x,t) \in G_h^1, \\ \Lambda_2^0 z^0(\xi,t) &\equiv \left\{\gamma^0(\xi,t,h_\xi,\tau)\delta_{\xi\bar{\xi}} - \delta_{\bar{t}}\right\}z^0(\xi,t) = 0, \qquad (\xi,t) \in G_{h\xi}^1, \end{aligned} \tag{3.2}$$

where $\gamma = \varepsilon^{-2}A$, $\gamma^0 = \gamma^0(\xi,t,h_\xi,\tau) = \gamma(\varepsilon\xi,t,\varepsilon h_\xi,\tau)$.

From condition (2.9) such a property of the function $\gamma^0(\xi,t,h_\xi,\tau)$ follows (we name it by the property ($\bigstar$)). Let (ξ_0,t_0), ξ_0, $t_0 > 0$ be some point from the neighbourhood of G_ξ^*, and also the set $\overline{G}_{0\xi} = [0,\xi_0] \times [t_0, t_0+\delta]$, $\delta > 0$ belong to $\overline{G}_\xi^*$. For any sufficiently small value $\delta_0 > 0$ one can find $\delta^0 = \delta^0(\delta_0)$-neighbourhood of a point $(0,t_0)$, belonging to $\overline{G}_{0\xi}$ (namely, the set $G_\xi^0 = (0,\xi^0) \times (t_0,t^0)$, where $\xi^0 = \delta^0$, $t^0 = t_0 + \delta^0$, $\overline{G}_\xi^0 \subseteq \overline{G}_{0\xi}$), such that for any (ξ,t_1), $(\xi,t_2) \in G_\xi^0$ and any h_ξ, $\tau \le m_1$, $m_1 = m_1(\delta_0)$, the following inequalities are satisfied

$$\begin{aligned} |\gamma^0(\xi,t_1,h_\xi,\tau) - 1| &\le m_2, \\ |\gamma^0(\xi,t_1,h_\xi,\tau) - \gamma^0(\xi,t_2,h_\xi,\tau)| &\le \delta_0, \qquad (\xi,t_1),(\xi,t_2) \in G_\xi^0. \end{aligned} \tag{3.3}$$

The following lemma is valid.

LEMMA 2. *Let for the solution of boundary value problem* (2.3) *the finite difference schemes with the grid approximation of form* (3.2) *be used, and assume for the coefficient* $\gamma^0(\xi, t, h_\xi, \tau)$ *that the property* ($\bigstar$) *is satisfied. Then there does not exist a finite difference scheme convergent* ε*-uniformly in the discrete maximum norm.*

Proof. To prove the lemma we estimate the functions $\omega^i(x,t) = u^i(x,t) - z^i(x,t)$, $i = 2,3$, where $u^i(x,t) = \Phi_0^i(x,t)$ is the solution of problem (2.3), $z^i(x,t)$ is the corresponding solution of the finite difference scheme; $\Phi_0^i(x,t) = \Phi^i(\varepsilon^{-1}x, t)$ are auxiliary functions. The functions $\Phi^i(\xi, t)$, $i = 0,1,2,3$, are defined by

$$\Phi^i(\xi,t) = i\int_0^t \Phi^{i-1}(\xi,t)\,d\tau, \qquad i = 1,2,3,\ \Phi^0(\xi,t) = 1 - v(\vartheta),$$

where $\vartheta = 2^{-1}\xi t^{-1/2}$ and $v(\vartheta)$ is the error function

$$v(\vartheta) = 2\pi^{-1/2}\int_0^\vartheta \exp(-\alpha^2)\,d\alpha.$$

The functions $\Phi^i(\xi,t)$ satisfy the differential equation $(\partial^2/\partial\xi^2 - \partial/\partial t)\Phi^i(\xi,t) = 0$, $(\xi,t) \in G_\xi$ and such relations on the boundary S_ξ: $\Phi^i(0,t) = t^i$, $t \in (0,T]$, $\Phi^i(\xi,0) = 0$, $0 < \xi < \infty$, $0 \le i \le 3$. We define the auxiliary "fitted coefficients" $\gamma^2(\xi,t,h_\xi,\tau)$, $\gamma^3(\xi,t,h_\xi,\tau)$ and the function $\gamma^*(\xi,t,h_\xi,\tau)$, i.e., the mean value of these coefficients, by

$$\begin{aligned}&\left(\gamma^i\delta_{\xi\bar\xi} - \delta_{\bar t}\right)\Phi^i(\xi,t) = 0, \qquad i = 2,3,\\ &\qquad\gamma^*(\xi,t,h_\xi,\tau) = 2^{-1}\left(\gamma^2(\xi,t,h_\xi,\tau) + \gamma^3(\xi,t,h_\xi,\tau)\right),\\ &\qquad\qquad (\xi,t) \in G_{h\xi}.\end{aligned} \tag{3.4}$$

For the functions $\Phi^i(\xi,t)$ and the coefficients $\gamma^j(\xi,t,h_\xi,\tau)$ the following are valid

$$\frac{\partial}{\partial t}\Phi^i(\xi,t) = i\Phi^{i-1}(\xi,t), \qquad (\xi,t) \in G_\xi,\ i = 1,2,3, \tag{3.5}$$

$$\Phi^j(\xi,t) = \Phi^j(0,t) + \frac{\partial}{\partial\xi}\Phi^j(0,t)\xi + 2^{-1}\frac{\partial^2}{\partial\xi^2}\Phi^j(0,t)\xi^2$$

$$+6^{-1}\frac{\partial^3}{\partial\xi^3}\Phi^j(\xi_1,t), \qquad 0 \le \xi_1 \le \xi,$$

$$\delta_{\xi\bar{\xi}}\Phi^j(\xi,t) = \frac{\partial^2}{\partial\xi^2}\Phi^j(\xi,t) + \frac{1}{4!}\frac{\partial^4}{\partial\xi^4}\Phi^j(\xi_2,t)h_\xi^2$$

$$+\frac{1}{4!}\frac{\partial^4}{\partial\xi^4}\Phi^j(\xi_3,t)h_\xi^2,$$

$$\xi - h_\xi \le \xi_2, \qquad \xi_3 \le \xi + h_\xi,$$

$$\delta_{\bar{t}}\Phi^j(\xi,t) = \frac{\partial}{\partial t}\Phi^j(\xi,t) + \frac{\partial^2}{\partial t^2}\Phi^j(\xi,t_1)\tau, \qquad t-\tau \le t_1 \le t,\ j = 2,3,$$

$$M^{-1}\Phi^i(0,t) \le \Phi^i(\xi,t) \le M\Phi^i(0,t), \qquad t \ge 2\tau,\ \xi^2 \le m\tau,\ i = 1,2,3,$$

$$M^{-1}\frac{\partial^{k+k_0}}{\partial\xi^k\,\partial t^{k_0}}\Phi^j(0,t) \le \frac{\partial^{k+k_0}}{\partial\xi^k\,\partial t^{k_0}}\Phi^j(\xi,t) \le M\frac{\partial^{k+k_0}}{\partial\xi^k\,\partial t^{k_0}}\Phi^j(0,t),$$

$$t \ge 2\tau,\ \xi^2 \le m\tau,\ j = 2,3,\ k + 2k_0 \le 4,$$

$$\left|\delta_{\bar{t}}\Phi^2(h_\xi,t) - 2\left[t - \frac{2}{\sqrt{\pi}}h_\xi t^{1/2} + 2^{-1}h_\xi^2\right]\right| \le M_1\left[\tau + h_\xi^3 t^{-1/2}\right],$$

$$\left|\delta_{\xi\bar{\xi}}\Phi^2(h_\xi,t) - 2\left[t - \frac{2}{\sqrt{\pi}}h_\xi t^{1/2} + \frac{7}{12}h_\xi^2\right]\right| \le M_2\left[\tau + h_\xi^3 t^{-1/2}\right],$$

$$\left|\delta_{\bar{t}}\Phi^3(h_\xi,t) - 3\left[t^2 + \frac{8}{3\sqrt{\pi}}h_\xi t^{3/2} + h_\xi^2 t\right]\right| \le M_3\left[\tau t + h_\xi^3 t^{1/2}\right],$$

$$\left|\delta_{\xi\bar{\xi}}\Phi^3(h_\xi,t) - 3\left[t^2 + \frac{8}{3\sqrt{\pi}}h_\xi t^{3/2} + \frac{7}{6}h_\xi^2 t\right]\right| \le M_4\left[\tau t + h_\xi^3 t^{1/2}\right],$$

$$\left|\gamma^2(\xi,t,h_\xi,\tau) - \left[1 - 12^{-1}h_\xi^2 t^{-1}\right]\right| \le M_5\left[\tau t^{-1} + h_\xi^3 t^{-3/2}\right],$$

$$\left|\gamma^3(\xi,t,h_\xi,\tau) - \left[1 - 6^{-1}h_\xi^2 t^{-1}\right]\right| \le M_6\left[\tau t^{-1} + h_\xi^3 t^{-3/2}\right],$$

$$\left|\gamma^2(\xi,t,h_\xi,\tau) - \gamma^3(\xi,t,h_\xi,\tau) - 12^{-1}h_\xi^2 t^{-1}\right| \le M_7\left[\tau t^{-1} + h_\xi^3 t^{-3/2}\right].$$

We see that the values γ^2 and γ^3 differ by a positive quantity of the order of the value $m_3 = 12^{-1}h_\xi^2 t^{-1}$ for sufficiently small values of τt^{-1} and $h_\xi t^{-1/2}$. Let us choose $h_{\xi 1}$ and τ_1, $h_{\xi 1} = h_{\xi 1}(t_0)$, $\tau_1 = \tau_1(h_{\xi 1})$ sufficiently small so that

$$M_8\left[\tau_1 t^{-1} + h_{\xi 1}^3 t^{-3/2}\right] \le 8^{-1}m_4, \qquad t = t_0,$$

where $m_4 = 12^{-1}h_{\xi 1}^2 t_0^{-1}$,

$$M_8 = (1 + T^2)\sum_{i=1}^{7} M_i.$$

The value δ^0 is chosen sufficiently small such that the inequality $\delta_0 \le 8^{-1} m_4$, inequalities (3.3), and the following inequality are true

$$\left|\gamma^i(\xi, t_1, h_\xi, \tau) - \gamma^i(\xi, t_2, h_\xi, \tau)\right| \le \delta_0,$$

$$(\xi, t_1), (\xi, t_2) \in \overline{G}_\xi^0, \ i = 2, 3.$$

The value $h_\xi = h_{\xi 2}$ is chosen to satisfy the condition $h_{\xi 2} \le m_4 12 t_0$, $h_{\xi 2} \le 2^{-1}\delta^0$ and further it is fixed.

We construct the set G_ξ^0

$$G_\xi^0 = \left\{(\xi, t) : 0 < \xi < 2h_{\xi 2}, t_0 < t \le t^0\right\}, \tag{3.7}$$

on which we shall study the fitted scheme. Note, the value $h_{\xi 2}$ and t_0, t^0 are independent of the parameter ε.

For $\xi = h_{\xi 2}$, $t = t_0$ at least one of the following inequalities is satisfied

$$\gamma^0\left(h_{\xi 2}, t_0, h_{\xi 2}, \tau\right) \ge \gamma^*\left(h_{\xi 2}, t_0, h_{\xi 2}, \tau\right), \tag{3.8a}$$

$$\gamma^0\left(h_{\xi 2}, t_0, h_{\xi 2}, \tau\right) \le \gamma^*\left(h_{\xi 2}, t_0, h_{\xi 2}, \tau\right). \tag{3.8b}$$

Suppose, that inequality (3.8a) is valid. Then

$$\gamma^0\left(h_{\xi 2}, t, h_{\xi 2}, \tau\right) \ge \gamma^3\left(h_{\xi 2}, t, h_{\xi 2}, \tau\right) + 8^{-1} m_5, \qquad (\xi, t) \in G_{h\xi}^0, \tag{3.9}$$

where $m_5 = 12^{-1} h_{\xi 2}^2 (t_0)^{-1}$. In this case for the function $\omega^{03}(\xi, t) = \omega^3(x(\xi), t)$ the relation is true

$$\Lambda_{(3.2)}^0 \omega^{03}(\xi, t) = \left(\gamma^0(\xi, t, h_{\xi 2}, \tau) - \gamma^3(\xi, t, h_{\xi 2}, \tau)\right)\delta_{\xi\bar{\xi}}\Phi^3(\xi, t)$$

$$\ge m_6^1, \qquad (\xi, t) \in G_{h\xi}^0. \tag{3.10a}$$

In the case of inequality (3.8b) for the function $\omega^{02}(\xi, t) = \omega^2(x(\xi), t)$ we have

$$\Lambda_{(3.2)}^0 \omega^{02}(\xi, t) \le -m_6^2, \qquad (\xi, t) \in G_{h\xi}^0, \tag{3.10b}$$

Let us assume that the finite difference scheme converges ε-uniformly

$$\left|\omega^i(x, t)\right| \le \lambda(h, \tau), \qquad (x, t) \in \overline{G}_h, \ \varepsilon \in (0, 1], \ i = 2, 3. \tag{3.11}$$

From here it follows that on the set $S_{h\xi}^0$ the following inequality is satisfied

$$\left|\omega^{0i}(\xi, t)\right| \le \lambda\left(\varepsilon h_{\xi 2}, \tau\right), \qquad (\xi, t) \in S_{h\xi}^0, \ i = 2, 3, \tag{3.12}$$

where $\lambda(\varepsilon h_{\xi 2}, \tau) \to 0$ for ε, $\tau \to 0$, $h_{\xi 2} = \text{const}$.

For the barrier function

$$w^0(\xi, t) = 2m_7\Big(2^{-1} + h_{\xi 2}^{-2}(\xi - h_{\xi 2})^2 - (\delta^0)^{-1}(t - t_0)\Big),$$
$$(\xi, t) \in \overline{G}_\xi^0 \quad (3.13)$$

the inequalities are valid

$$\Lambda^0_{(3.2)} w^0(\xi, t) \le 2m\Big(2\big(1 + m_{2(3.3)}\big)h_{\xi 2}^{-2} + (\delta^0)^{-1}\Big), (\xi, t) \in G^0_{h\xi},$$
$$w^0(\xi, t) \ge m, \qquad (\xi, t) \in S^0_{h\xi},\ m = m_7.$$

Taking into account relation (3.12) under suitable choice of $m_{(3.13)}$ it is shown using the maximum principle that, at least for one value $i = j$, the next inequality is true

$$\Big| u^{0j}\big(\varepsilon h_{\xi 2}, t^0\big) - z^{0j}\big(\varepsilon h_{\xi 2}, t^0\big)\Big| \ge m \qquad \text{for } \tau \le \tau_1,\ \varepsilon \in (0, \varepsilon_1], \quad (3.14)$$

where τ_1, ε_1 are sufficiently small numbers. From (3.14) it follows that

$$\max_{\overline{G}_h} \big| u^j(x, t) - z^j(x, t)\big| \ge m \quad (3.15)$$

for any h, $\tau \le m_1$, $h \le \varepsilon_1 h_{\xi 2}$, $\varepsilon_1 = \varepsilon_{1(3.14)}$, $\tau \le \tau_{1(3.14)}$; $h = \varepsilon h_{\xi 2}$. Inequality (3.15) contradicts (3.11) for $\varepsilon \in (0, \varepsilon_1]$. This completes the proof of Lemma 2.

The proof of Theorem 1 is similar to the proof of Lemma 2. Indeed, let $z(x, t)$ and $z^0(\xi, t)$ be solutions of the equations

$$\Lambda_{(3.16)} z(x, t) \equiv \big\{\varepsilon^2 \gamma(x, t, h, \tau, \varepsilon)\delta_{x\bar{x}} + B(x, t, h, \tau, \varepsilon)\delta_{\tilde{x}} - \delta_{\bar{t}}\big\} z(x, t)$$
$$= 0, \qquad (x, t) \in G_h,$$
$$\Lambda^0_{(3.16)} z^0(\xi, t) \equiv \big\{\gamma^0(\xi, t, h_\xi, \tau)\delta_{\xi\bar{\xi}} + b(\xi, t, h_\xi, \tau)\delta_{\tilde{\xi}} - \delta_{\bar{t}}\big\} z^0(\xi, t)$$
$$= 0, \qquad (\xi, t) \in G_{h\xi}, \quad (3.16)$$

where $b(\xi, t, h_\xi, \tau) = B_0^0(\xi, t, h_\xi, \tau)$. From condition (2.9) it follows that the value of b becomes arbitrarily small for $(\xi, t) \in G_\xi^*$, h_ξ, $\tau \le m$, if m is a sufficiently small number. With this argument the expressions $\Lambda^0_{(3.16)}\omega^{0i}(\xi, t)$, $\Lambda^0_{(3.16)} w^0(\xi, t)$ slightly differ from the expressions $\Lambda^0_{(3.2)}\omega^{0i}(\xi, t)$, $\Lambda^0_{(3.2)} w^0(\xi, t)$, $i = 2, 3$. The proof of Theorem 1 repeats the proof of Lemma 2 with obvious adjustments.

Notation. $L_{(j,k)}$ or $f_{(j,k)}$ will denote that this operator or function is introduced in the text in $(j.k)$.

4. FITTED OPERATOR METHODS WITHOUT A DISCRETE MAXIMUM PRINCIPLE

Some methods for the construction of grid approximations, for example, the finite element method, generate finite difference schemes, which do not inherit the maximum principle even if the maximum principle for a boundary value problem is satisfied. In connection with this such a question appears. Among the difference schemes for which the maximum principle, generally speaking, is not valid, can we find an ε-uniformly convergent scheme based on the fitted operator method?

We shall describe a class of finite difference schemes, that we name class B, differing from class A, in which the fitted operator scheme for problem (2.3) is to be constructed.

On the set $\overline{G}$ we introduce the uniform grid $\overline{G}_h^u$. On the grid G_h^u we consider the grid equation

$$\Lambda_{(4.1)} z(x,t) \equiv \left\{\varepsilon^2 \gamma(x,t,h,\tau)\,\delta_{x\bar{x}} - \delta_{\bar{t}}\right\} z(x,t) = 0, \qquad (x,t) \in G_h, \tag{4.1a}$$

approximating Eq. (2.3a). For variables ξ, t this equation takes the form

$$\Lambda^0_{(4.1)} z^0(\xi,t) \equiv \left\{\gamma^0(\xi,t,h_\xi,\tau)\,\delta_{\xi\bar{\xi}} - \delta_{\bar{t}}\right\} z^0(\xi,t) = 0, \qquad (\xi,t) \in G_{h\xi}, \tag{4.1b}$$

where $\gamma^0(\xi,t,h_\xi,\tau) = \gamma(\varepsilon\xi,t,\varepsilon h_\xi,\tau)$. For simplicity we consider the case when the coefficient γ^0 does not depend on the parameter ε. The fulfillment of condition (2.9) is not assumed. In particular, the coefficient γ^0 can take negative values also. Suppose that the coefficient γ^0 is bounded

$$\left|\gamma^0(\xi,t,h_\xi,\tau)\right| \le M, \qquad (\xi,t) \in G_{h\xi}. \tag{4.1c}$$

In this class of difference schemes, an ε-uniformly convergent scheme is to be constructed.

The following theorem is valid.

THEOREM 2. *In the class of difference schemes B there does not exist a finite difference scheme, whose solution as $h, \tau \to 0$ converges pointwise to the solution of the boundary value problem* (2.3) *ε-uniformly.*

Proof. The proof of the theorem is again by contradiction. Assume that on the uniform grid $\overline{G}_h^u$ the solution of the finite difference scheme with grid equations (4.1) converges to the solution of the boundary value problem (2.3) ε-uniformly. Let the function $z^i(x,t)$ be the solution of the grid problem related to problem (2.3) for $u(x,t) \equiv u^i(x,t) = \Phi_0^i(x,t)$, $i = 2,3$. Denote by $\omega^i(x,t) = u^i(x,t) - z^i(x,t)$, $i = 2,3$.

On the segment $[0,T]$ we choose the set $[t_0, t^0]$ where $t_0 > 0$. Let m^1, m^2 be sufficiently small numbers and $h_{\xi 1} = m_1$, $\tau \le m^2$. Define the set G^0_ξ

$$G^0_\xi = \left\{ (\xi, t) : 0 < \xi < 2h_{\xi 1},\ t_0 < t \le t^0 \right\}. \tag{4.2}$$

Let for this set the set G^0 in variables x, t be related. On the set $\overline{G}^0$ the grid $\overline{G}^0_h = \overline{G}^0 \cap \overline{G}_h$ is defined. We shall consider the functions $\omega^i(x,t)$ for (x,t) for $(x,t) \in \overline{G}^0_h$. The functions $\omega^i(x,t)$ and $\omega^{0i}(\xi, t)$ are solutions of the next problems

$$\begin{aligned} \Lambda_{(4.1)} \omega^i(x,t) &= F^i_{(4.3)}(x,t), & (x,t) &\in G^0_h, \\ \omega^i(x,t) &= \varphi^i_{(4.3)}(x,t), & (x,t) &\in S^0_h,\ i = 2,3; \end{aligned} \tag{4.3}$$

$$\begin{aligned} \Lambda^0_{(4.1)} \omega^{0i}(\xi,t) &= F^i_{(4.4)}(\xi,t), & (\xi,t) &\in G^0_{h\xi}, \\ \omega^{0i}(\xi,t) &= \varphi^i_{(4.4)}(\xi,t), & (\xi,t) &\in S^0_{h\xi},\ i = 2,3. \end{aligned} \tag{4.4}$$

Here

$$F^i_{(4.3)}(x,t) = \Lambda_{(4.1)} \Phi^i_0(x,t),$$

$$\varphi^i_{(4.3)}(x,t) = \Phi^i_0(x,t) - z^i(x,t), \qquad (x,t) \in S^0_h,$$

$$\begin{aligned} F^i_{(4.4)}(\xi,t) &= \Lambda^0_{(4.1)} \Phi^i(\xi,t) \\ &= \left[\gamma^0(\xi,t,h_\xi,\tau) - \gamma^i(\xi,t,h_\xi,\tau) \right] \delta_{\xi\bar{\xi}} \Phi^i(\xi,t), \end{aligned}$$

$$\varphi^i_{(4.4)}(\xi,t) = \Phi^i(\xi,t) - z^{0i}(\xi,t), \qquad (\xi,t) \in S^0_{h\xi},$$

$$\gamma^i(\xi,t,h_\xi,\tau) = \gamma^i_{(3.4)}(\xi,t,h_\xi,\tau).$$

Instead of problem (4.3) we shall consider the auxiliary problem

$$\begin{aligned} \Lambda_{(4.1)} \omega^i(x,t) &= F^i_{(4.3)}(x,t), & (x,t) &\in G^0_h, \\ \omega^i(x,t) &= 0, & (x,t) &\in S^0_h,\ i = 2,3. \end{aligned} \tag{4.5}$$

To solve problem (4.5) for the function $\omega^i(x,t)$ brings us to the following boundary value problem

$$\begin{aligned} L_{(2.1)} u(x,t) &= 0, & (x,t) &\in G^0, \\ u(x,t) &= \Phi^i_0(x,t), & (x,t) &\in S^0,\ i = 2,3 \end{aligned} \tag{4.6}$$

and the corresponding difference scheme

$$\begin{aligned} \Lambda_{(4.1)} z(x,t) &= 0, && (x,t) \in G_h^0, \\ z(x,t) &= \Phi_0^i(x,t), && (x,t) \in S_h^0,\ i = 2,3. \end{aligned} \tag{4.7}$$

Here $\omega^i(x,t) = u^i(x,t) - z^i(x,t)$, $i = 2,3$.

The following statement is true.

LEMMA 3. *The solution of the difference scheme* (4.7) *does not converge in the discrete maximum norm to the solution of boundary value problem* (4.6) *ε-uniformly.*

Proof. Assume that the solution of the discrete problem (4.7) converges ε-uniformly to the solution of the boundary value problem (4.6). Making such an assumption, we are led to a contradiction.

In the variables ξ, t the problem (4.5) takes the form

$$\begin{aligned} \Lambda_{(4.1)}^0 \omega^{0i}(\xi,t) &= F_{(4.4)}^i(\xi,t), && (\xi,t) \in G_{h\xi}^0, \\ \omega^{0i}(\xi,t) &= 0, && (\xi,t) \in S_{h\xi}^0,\ i = 2,3. \end{aligned} \tag{4.4}$$

Let us introduce the function $v^i(t) = \omega^{0i}(h_{\xi 1}, t)$. This function is the solution of the discrete Cauchy problem

$$\begin{aligned} &\Lambda_{(4.8)} v^i(t) \equiv \{\delta_{\bar t} + c(t)\} v^i(t) = -F_{(4.4)}^i(\xi,t), \\ &\xi = h_{\xi_1}, \qquad t \in \omega_0^1, \qquad v^i(t_0) = 0,\ i = 2,3. \end{aligned} \tag{4.8}$$

Here $\overline{\omega}_0^1$ is a uniform grid on the segment $[t_0, t^0]$, $\overline{\omega}_0^1 = \overline{\omega}_0 \cap [t_0, t^0]$, $\overline{\omega}_0 = \overline{\omega}_{0(2.4)}$ is a grid on the segment $[0,T]$, $c(t) = 2h_{\xi 1}^{-2}\gamma^0(h_{\xi 1}, t, h_{\xi 1}, \tau)$.

Let the function $g(t)$ be the solution of the problem

$$\delta_{\bar t} g(t) = c(t) g(t-\tau), \qquad t \in \omega_0^1,\ g(t_0) = 1.$$

The value τ is assumed to be sufficiently small and satisfies the condition

$$\tau \le 4^{-1} M_{(4.1)}^{-1} h_{\xi 1}^2.$$

Then for the function $g(t)$ the following estimate is fulfilled

$$g(t) \ge m \exp\left(-M h_{\xi 1}^{-2}\right), \qquad t \in \overline{w}_0^1. \tag{4.9}$$

The function $w^i(t) = g(t)v^i(t)$, $i = 2, 3$, is the solution of the problem

$$\begin{aligned}\delta_{\bar{t}} w^i(t) &= -g(t-\tau)F^i(h_{\xi 1}, t)\\ &= g(t_k - \tau)\big[\gamma^i(h_{\xi 1}, t_k, h_{\xi 1}, \tau) - \gamma^0(h_{\xi 1}, t_k, h_{\xi 1}, \tau)\big]\\ &\quad\times \delta_{\xi\bar{\xi}}\Phi^i(h_{\xi 1}, t_k), \qquad t_k \in \omega_0^1,\ w^i(t_0) = 0,\ i = 2, 3,\end{aligned}$$

where $t_k = t_0 + k\tau$, $k = 0, 1, \dots, K$, $t_K = t^0$. Eliminating $\gamma^0(h_{\xi 1}, t, h_{\xi 1}, \tau)$, we come to the relations

$$\begin{aligned}&\delta_{\xi\bar{\xi}}\Phi^3(h_{\xi 1}, t)\delta_{\bar{t}} w^2(t) - \delta_{\xi\bar{\xi}}\Phi^2(h_{\xi 1}, t)\delta_{\bar{t}} w^3(t)\\ &\quad= g(t-\tau)\,\delta_{\xi\bar{\xi}}\Phi^2(h_{\xi 1}, t)\delta_{\xi\bar{\xi}}\Phi^3(h_{\xi 1}, t)\\ &\qquad\times\big[\gamma^2(h_{\xi 1}, t, h_{\xi 1}, \tau) - \gamma^3(h_{\xi 1}, t, h_{\xi 1}, \tau)\big],\\ &\qquad\qquad t \in \omega_0^1,\ w^2(t_0) = w^3(t_0) = 0.\end{aligned}$$

"Integrating" with respect to t we find

$$\begin{aligned}&\delta_{\xi\bar{\xi}}\Phi^3(h_{\xi 1}, t_k)w^2(t_k) - \delta_{\xi\bar{\xi}}\Phi^2(h_{\xi 1}, t_k)w^3(w_k)\\ &\quad-\tau\sum_{s=1}^{k}\delta_{\bar{t}}\,\delta_{\xi\bar{\xi}}\Phi^3(h_{\xi 1}, t_s)w^2(t_{s-1}) + \tau\sum_{s=1}^{k}\delta_{\bar{t}}\,\delta_{\xi\bar{\xi}}\Phi^2(h_{\xi 1}, t_s)w^3(t_{s-1})\\ &\quad= \tau\sum_{s=1}^{k} g(t_{s-1})\,\delta_{\xi\bar{\xi}}\Phi^2(h_{\xi 1}, t_s)\delta_{\xi\bar{\xi}}\Phi^3(h_{\xi 1}, t_s)\\ &\qquad\times\big[\gamma^2(h_{\xi 1}, t_s, h_{\xi 1}, \tau) - \gamma^3(h_{\xi 1}, t_s, h_{\xi 1}, \tau)\big], \qquad t_k \in \omega_0^1. \quad (4.10)\end{aligned}$$

According to the assumption the functions $z^i(x, t)$ as $h, \tau \to 0$ converge to the functions $\Phi_0^i(x, t)$ ε-uniformly. Taking into account the properties of the function $c(t)$ and relations (3.5), (4.9), (4.1c), Eq. (4.10) brings us to the inequality

$$\lambda(h, \tau) \geq m h_{\xi 1}^{-2}\exp\big(-M h_{\xi 1}^2\big)(t - t_0) - M\tau, \qquad t \in \big[t_0, t^0\big],$$

where $h = \varepsilon h_{\xi 1}$. This inequality is contradictory for sufficiently small values τ and ε. Consequently, the functions $z^i(x, t)$, $i = 2, 3$, do not converge ε-uniformly. The lemma is proved.

Returning to the proof of Theorem 2 we note that the proof is similar to the proof of Lemma 3. Assuming that the approximate solution converges ε-uniformly on $\overline{G}_h$ causes ε-uniform closeness of functions $z^i(x, t)$ and $u^i(x, t)$, $i = 2, 3$, on the boundary S_h^0 of the set G_h^0. As in the proof of lemma, we come to a contradiction. This completes the proof of the theorem.

Remark 4.1. By a similar way it is established that in the considered class B of difference schemes, in which Eqs. (4.1) are replaced by Eqs. (3.16), where the coefficients $\gamma^0(\xi, t, h_\xi, \tau)$ and $b(\xi, t, h_\xi, \tau)$ are bounded, there do not exist difference schemes convergent ε-uniformly.

Remark 4.2. The statement of Theorem 2 and its proof remain valid also in the case when the coefficients γ^0 and b in (3.16) depend on the parameter ε, $\gamma^0 = \gamma^0(\xi, t, h_\xi, \tau, \varepsilon)$, $b = b(\xi, t, h_\xi, \tau, \varepsilon)$, and also condition (4.1c) is replaced by the condition

$$\left|\gamma^0(\xi, t, h_\xi, \tau, \varepsilon)\right|, \left|b(\xi, t, h_\xi, \tau, \varepsilon)\right| \le M, \qquad (\xi, t) \in G_{h\xi}.$$

Remark 4.3. Let the fitted operator scheme be constructed in the class of finite difference schemes on stencils of explicit four-point schemes or weighted implicit six-point schemes, and in the variables ξ, t the coefficients multiplying the difference space derivatives are bounded uniformly with respect to the parameter ε. In this class of schemes there does not exist also a scheme convergent ε-uniformly.

Remark 4.4. It is possible to consider the construction of the scheme on the base of the Rothe method (i.e., method of lines) with "grid" equations

$$\Lambda v(x,t) \equiv \left\{\varepsilon^2 \gamma(x,t,h)\,\delta_{x\bar{x}} - \frac{\partial}{\partial t}\right\} v(x,t) = 0, \qquad x \in \omega_1,\ t \in (0,T],$$

$\omega_1 = \omega_{1(2.4)}$, approximating Eq. (2.3a); the coefficient $\gamma(x, t, h)$ is supposed to be bounded uniformly with respect to the parameter. Notice that only the space variable has been discretized. In this class of schemes there does not exist a scheme convergent ε-uniformly, that is, a scheme for whose solution such an estimate is valid

$$\max_{t \in [0,T],\, x \in \overline{\omega}_1} \left|u(x,t) - v(x,t)\right| \le \lambda(h),$$

where $\lambda(h) \to 0$ uniformly with respect to the parameter for $h \to 0$. This is proved in an analogous fashion to the above proof.

5. ELLIPTIC EQUATION WITH A PARABOLIC BOUNDARY LAYER

The investigation of heat and diffusion processes in moving media brings us to singularly perturbed elliptic and parabolic equations with convective terms. When the parameter tends to zero, in the neighbourhood of sides along which the movement of the media occurs, parabolic boundary layers appear.

Let us consider the following example. Let on a unit square

$$D = \{x : 0 < x_1, x_2 < 1\} \tag{5.1}$$

the Dirichlet problem for the equation of elliptic type

$$L_{(5.2)}u(x) \equiv \left\{\varepsilon^2\Delta + \frac{\partial}{\partial x_1}\right\}u(x) = f(x), \qquad x \in D, \tag{5.2a}$$

$$u(x) = \varphi(x), \qquad x \in \Gamma. \tag{5.2b}$$

be considered. Here $\Gamma = \overline{D} \setminus D$ is the boundary of the domain D, Δ is the Laplace operator, and $f(x)$, $\varphi(x)$ are sufficiently smooth functions.

The characteristics of the reduced equation (Eq. (5.2a) for $\varepsilon = 0$) are defined by the vector $(-1, 0)$. Depending on the disposition of the sides of the square D with respect to the vector $(-1, 0)$, the boundary Γ is split into the subsets Γ^+, Γ^-, and Γ^0 (these subsets are suppose to be closed). The set $\Gamma^- = \{x : x_1 = 0\}$ (the set $\Gamma^+ = \{x : x_1 = 1\}$) is such part of the boundary Γ, through which the characteristics leave (enter into) the set D. The set $\Gamma^0 = \{x : x_2 = 0, 1\}$ is formed by characteristics of the reduced equation.

When the parameter tends to zero, elliptic and parabolic layers appear respectively in the neighbourhood of the sets Γ^- and Γ^0. In the neighbourhood of the set Γ^+ boundary layers do not appear.

We are interested in grid approximations of the boundary value problem (5.2) on rectangular grids. Let

$$\overline{D}_h = \overline{\omega}_1 \times \overline{\omega}_2 \tag{5.3}$$

be a grid on $\overline{D}$, where $\overline{\omega}_s$, generally speaking, is a uniform grid on the segment $[0, 1]$. By $N_s + 1$ we denote the number of nodes in the grid $\overline{\omega}_s$, h_s is the maximal step-size, $h_s \le MN_s^{-1}$. Let $z(x)$, $x \in \overline{D}_{h(5.3)}$ be a solution of some finite difference scheme. We say this solution converges ε-uniformly, if for the function $z(x)$ the following estimate is valid,

$$\max_{\overline{D}_h} |u(x) - z(x)| \le \lambda\left(N_1^{-1}, N_2^{-1}\right),$$

where for $N_1, N_2 \to \infty$ the value $\lambda(N_1^{-1}, N_2^{-1}) \to 0$ tends to zero uniformly with respect to the parameter ε.

For problem (5.2) we wish to construct a scheme based on the fitted operator method, convergent ε-uniformly.

We give the problem formulation for a boundary value problem and describe a class of difference schemes (named class C), in which we shall seek to construct fitted schemes convergent ε-uniformly.

Let the right-hand side $f(x)$ and the boundary function $\varphi(x)$ satisfy the condition

$$f(x) \equiv 0, \qquad x \in \overline{D},$$

$$\varphi(x) = \begin{cases} \varphi_0(x_1), & x \in \Gamma, \quad x_2 = 0, \\ 0, & x \in \Gamma, \quad x_2 \neq 0. \end{cases} \tag{5.4}$$

For this case the solution of problem (5.2) is singular.

The principal part in the asymptotic expansion of the solution on the set

$$\overline{D}^{\delta} = \{x : \delta \le x_1 \le 1, 0 \le x_2 \le 1\},$$

where $\delta > 0$ is a small number, is the solution of the equation of parabolic type

$$\begin{gathered} L_{(5.5)}u(x) \equiv \left\{ \varepsilon^2 \frac{\partial^2}{\partial x_2^2} + \frac{\partial}{\partial x_1} \right\} u(x) = 0, \qquad x \in G^{\delta}, \\ u(x) = \varphi(x), \qquad x \in S^{\delta}. \end{gathered} \tag{5.5}$$

Here

$$G^{\delta} = \{x : \delta \le x_1 < 1, 0 < x_2 < 1\}, \qquad S^{\delta} = \overline{G}^{\delta} \setminus G^{\delta},$$

the variable $y_1 = 1 - x_1$, $0 < y_1 \le 1 - \delta$ is used as a time variable. For the solution of problem (5.2), (5.4) the following estimate is fulfilled

$$\left| u(x) - u_{(5.5)}(x) \right| \le M\varepsilon^2, \qquad x \in \overline{D}^{\delta},$$

with $u(x)$ and $u_{(5.5)}(x)$, the solutions of problems (5.2), (5.4), and (5.5), respectively.

Let us describe the class C. On the set $\overline{D}$ we introduce the uniform rectangular grids

$$\overline{D}_h = \overline{\omega}_1 \times \overline{\omega}_2, \tag{5.6}$$

where $\overline{\omega}_s$ is a uniform grid on the segment $[0, 1]$ on the axis x_s with the step-size h_s, $s = 1, 2$.

We say, in the case of the grid $\overline{D}_{h(5.6)}$, the solution $z(x)$ of the difference scheme converges ε-uniformly, if the following estimate is valid

$$\max_{\overline{D}_h} \left| u(x) - z(x) \right| \le \lambda(h_1, h_2),$$

where $\lambda(h_1, h_2) \to 0$ uniformly with respect to the parameter ε as $h_1, h_2 \to 0$.

On the grid $\overline{D}_h$, Eqs. (5.2a), (5.4) are approximated by the finite difference scheme on the five point stencil in such a form

$$\Lambda_{(5.7)} z(x) \equiv \left\{ \varepsilon^2 \sum_{s=1,2} \gamma_s(x, h_1, h_2)\, \delta_{xs\overline{xs}} + \delta_{\overline{x1}} \right\} z(x) = 0, \qquad x \in D_h. \tag{5.7a}$$

From the variables x_1, x_2 we pass to the variables ξ_1, ξ_2, $\xi = (\xi_1, \xi_2)$, $\xi_1 = 1 - x_1$, $\xi_2 = \varepsilon^{-1} x_2$. In the new variables Eqs. (5.2a), (5.4), (5.7a) take the form

$$L^0_{(5.2)} u^0(\xi) \equiv \left\{ \varepsilon^2 \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} - \frac{\partial}{\partial \xi_1} \right\} u^0(\xi) = 0, \qquad \xi \in D_\xi,$$

$$\Lambda^0_{(5.7)} z^0(\xi) \equiv \left\{ \varepsilon^2 \gamma^0_1(\xi, h_{\xi 1}, h_{\xi 2})\, \delta_{\xi 1 \overline{\xi 1}} + \gamma^0_2(\xi, h_{\xi 1}, h_{\xi 2})\, \delta_{\xi 2 \overline{\xi 2}} - \delta_{\overline{\xi 1}} \right\} z^0(\xi)$$
$$= 0, \qquad \xi \in D_{h\xi}. \tag{5.7b}$$

Here $v^0(\xi) = v(x(\xi)) = v(1 - \xi_1, \varepsilon \xi_2)$, $\gamma_s^0(\xi, h_{\xi 1}, h_{\xi 2}) = \gamma_2(x(\xi), h_{\xi 1}, \varepsilon h_{\xi 2})$, $h_{\xi 1} = h_1$, $h_{\xi 2} = \varepsilon^{-1} h_2$; in the new variables the sets D^*_ξ correspond to the sets $D^* \subseteq \overline{D}$. The coefficients γ_s^0 are supposed to be bounded

$$\left| \gamma_s^0(\xi, h_{\xi 1}, h_{\xi 2}) \right| \le M, \qquad \xi \in D_{h\xi},\ s = 1, 2. \tag{5.7c}$$

In this class of finite difference schemes we wish to construct a scheme convergent ε-uniformly.

The following theorem is valid.

THEOREM 3. *In the class of difference schemes C there does not exist a finite difference scheme, whose solution as* $h_1, h_2 \to 0$ *converges pointwise to the solution of the boundary problem* (5.2), (5.4) ε*-uniformly.*

Proof. The proof of the theorem is carried out in a similar manner to the proof of Theorem 2. Assume that an ε-uniformly convergent scheme in the class C exists. Let us study this scheme.

On the segment $[0, 1]$ on the axis ξ_1 we choose the set $[\xi_{10}, \xi_1^0]$, where $\xi_{10}, \xi_1^0 \in (0, 1)$. We shall consider the set

$$D^0_\xi = \left\{ \xi : 0 < \xi_2 < 2h_{\xi 2},\ \xi_{10} < \xi \le \xi_1^0 \right\}. \tag{5.8}$$

On the set $\overline{D}^0_{h\xi(5.8)} = \overline{D}^0_{\xi(5.8)} \cap \overline{D}_{h\xi}$ we estimate the function $\omega(x) = u(x) - z(x)$ for $u(x) = \Phi_0^i(x_2, 1 - x_1)$, $i = 2, 3$. Denote by $\omega^i(x) = \Phi_0^i(x_2, 1 - x_1) - z^i(x)$, where $z^i(x)$ is the solution of the grid problem corresponding to the function $u(x) = \Phi_0^i(x_2, 1 - x_1)$. The functions $\omega^{0i}(\xi) = \omega^i(x(\xi))$

are the solutions of problems

$$\begin{aligned} \Lambda^0_{(5.7)}\omega^{0i}(\xi) &= F^i(\xi), \qquad \xi \in D^0_{h\xi}, \\ \omega^{0i}(\xi) &= \varphi^i(\xi), \qquad \xi \in S^0_{h\xi},\ i = 2,3. \end{aligned} \tag{5.9}$$

Here

$$\begin{gathered} F^i(\xi) = \Big\{\varepsilon^2\gamma_1^0(\xi, h_{\xi 1}, h_{\xi 2})\delta_{\xi 1\overline{\xi 1}} + \gamma_2^0(\xi, h_{\xi 1}, h_{\xi 2})\delta_{\xi 2\overline{\xi 2}} \\ -\gamma^i(\xi_2, \xi_1, h_{\xi 2}, h_{\xi 1})\delta_{\xi 2\overline{\xi 2}}\Big\}\Phi^i(\xi_2, \xi_1), \\ \varphi^i(\xi) = \Phi^i(\xi_2, \xi_1) - z^i(\xi), \\ \xi \in S^0_{h\xi}, \quad \gamma^i(\xi_2, \xi_1, h_{\xi 2}, h_{\xi 1}) = \gamma^i_{(3.4)}(\xi_2, \xi_1, h_{\xi 2}, h_{\xi 1}). \end{gathered}$$

The problem (5.9) is written in such a form

$$\begin{gathered} \Lambda_{(5.10)}v^i(\xi_1) \equiv \Big\{\delta_{\overline{\xi 1}} + c(\xi_1)\Big\}v^i(\xi_1) = F^{1i}_{(5.10)}(\xi) - F^{2i}_{(5.10)}(\xi), \\ \xi_2 = h_{\xi 2}, \qquad \xi_1 \in \omega^1_{1\xi}, \qquad v^i(\xi_{10}) = \varphi^i(\xi_{10}, h_{\xi 2}),\ i = 2,3. \end{gathered} \tag{5.10}$$

Here $v^i(\xi_1) = \omega^{0i}(\xi_1, h_{\xi 2})$, $\overline{\omega}^1_{1\xi}$ is a uniform grid on the segment $[\xi_{10}, \xi_1^0]$ generated by the grid $\overline{\omega}_{1\xi}$, $\overline{\omega}_1 = \overline{\omega}_{1(5.6)}$; $c(\xi_1) = 2h^{-2}_{\xi 2}\gamma_2^0(\xi_1, h_{\xi 2}, h_{\xi 1}, h_{\xi 2})$,

$$\begin{gathered} F^{1i}_{(5.10)}(\xi) = \varepsilon^2\gamma_1^0(\xi, h_{\xi 1}, h_{\xi 2})\delta_{\xi 1\overline{\xi 1}}\omega^{0i}(\xi) + h^{-2}_{\xi 2}\gamma_2^0(\xi, h_{\xi 1}, h_{\xi 2}) \\ \times\big[\varphi^i(\xi_1, 0) + \varphi^i(\xi_1, 2h_{\xi 2})\big] \\ - \varepsilon^2\gamma_1^0(\xi, h_{\xi 1}, h_{\xi 2})\delta_{\xi 1\overline{\xi 1}}\Phi^i(\xi_2, \xi_1), \\ F^{2i}_{(5.10)}(\xi) = \big[\gamma_2^0(\xi, h_{\xi 1}, h_{\xi 2}) - \gamma^i(\xi_2, \xi_1, h_{\xi 2}, h_{\xi 1})\big]\delta_{\xi 2\overline{\xi 2}}\Phi^i(\xi_2, \xi_1), \\ \gamma^i(\xi_2, \xi_1, h_{\xi 2}, h_{\xi 1}) = \gamma^i_{(3.4)}(\xi_2, \xi_1, h_{\xi 2}, h_{\xi 1}). \end{gathered}$$

Because of the assumption that the functions $z^i(x)$ converge ε-uniformly, the following estimates are true

$$\begin{gathered} \big|v^i(\xi_{10})\big| \le \lambda(h_1, h_2), \\ \big|c(\xi_1)\big| \le Mh^{-2}_{\xi 2}, \qquad \xi_1 \in \overline{\omega}^1_{1\xi}, \\ \Big|F^{1i}_{(5.10}(\xi_1, h_{\xi 2})\Big| \le M\Big\{\big[\varepsilon^2 h^{-2}_{\xi 1} + h^{-2}_{\xi 2}\big]\lambda(h_1, h_2) + \varepsilon^2\Big\}, \qquad \xi_1 \in \overline{\omega}^1_{1\xi}. \end{gathered} \tag{5.11}$$

The function $F^{2i}_{(5.10)}(\xi)$ depends on the coefficient γ_2^0. Using Eqs. (5.10) for $i = 2$ and 3 we eliminate the coefficient γ_2^0:

$$\begin{aligned}
&\delta_{\xi 2\overline{\xi 2}}\Phi^3(\xi_2, \xi_1)\{\delta_{\overline{\xi 1}} + c(\xi_1)\}v^2(\xi_1) \\
&\quad - \delta_{\xi 2\overline{\xi 2}}\Phi^2(\xi_2, \xi_1)\{\delta_{\overline{\xi 1}} + c(\xi_1)\}v^3(\xi_1) \\
&\quad - \delta_{\xi 2\overline{\xi 2}}\Phi^3(\xi_2, \xi_1)F^{12}_{(5.10)}(\xi) + \delta_{\xi 2\overline{\xi 2}}\Phi^2(\xi_2, \xi_1)F^{13}_{(5.10)}(\xi) \\
&\quad = \delta_{\xi 2\overline{\xi 2}}\Phi^2(\xi_2, \xi_1) \\
&\qquad \times \delta_{\xi 2\overline{\xi 2}}\Phi^3(\xi_2, \xi_1)\left[\gamma^3(\xi_2, \xi_1, h_{\xi 2}, h_{\xi 1}) - \gamma^2(\xi_2, \xi_1, h_{\xi 2}, h_{\xi 1})\right], \\
&\qquad\qquad \xi_1 \in \omega^1_{1\xi},\ \xi_2 = h_{\xi 2}.
\end{aligned}$$

This equation is “integrated” with respect tot ξ_1. Taking into account estimates (5.11) and estimating the left part of the relation from above and the right part from below, we come to the inequality

$$M\left\{\left[\varepsilon^2 h_{\xi 1}^{-2} + h_{\xi 2}^{-2}\right]\lambda\left(h_{\xi 1}, \varepsilon h_{\xi 2}\right) + \varepsilon^2\right\} \geq m. \tag{5.12}$$

We choose the value $h_{\xi 2}$ sufficiently small and then fix it. Further we choose $\varepsilon = \varepsilon(h_{\xi 1}) = h_{\xi 1}$ and then the value $h_{\xi 1}$ tends to zero. For sufficiently small values of $h_{\xi 1}$ the inequality (5.12) is a contradiction.

Consequently, the assumption on ε-uniform convergence of functions $z^i(x)$ is incorrect. This completes the proof of Theorem 3.

Remark 5.1. The statement of the theorem remains valid also in the case when the coefficients γ_s^0 depend on the parameter ε.

Remark 5.2. We can consider schemes based on Rothe’s method (i.e., method of lines) with “grid” equations

$$\Lambda v(x) \equiv \left\{\varepsilon^2\gamma_1(x, h_2, \varepsilon)\frac{\partial^2}{\partial x_1^2} + \varepsilon^2\gamma_2(x, h_2, \varepsilon)\,\delta_{x2\overline{x2}} + \frac{\partial}{\partial x_1}\right\}v(x) = 0,$$

$$x = (x_1, x_2), \qquad x_2 \in \omega_2,\ x_1 \in (0,1),\ \omega_2 = \omega_{2(5.6)},$$

approximating Eqs. (5.2a), (5.4). The coefficients $\gamma_s^0(\xi, h_{\xi 2}, \varepsilon) = \gamma_s(x(\xi), \varepsilon h_{\xi 2}, \varepsilon)$, $\xi = (\xi_1, \xi_2)$, $\xi_1 = 1 - x_1$, $\xi_2 = \varepsilon^{-1}x_2$ are supposed to be bounded uniformly with respect to the parameter. In such a class of schemes there does not exist a scheme convergent ε-uniformly, that is, scheme for whose solution the next estimate is fulfilled

$$\max_{x_1 \in [0,1],\, x_2 \in \overline{\omega}_2} \left|u(x) - v(x)\right| \leq \lambda(h_2),$$

where $\lambda(h_2) \to 0$ uniformly with respect to the parameter ε for $h_2 \to 0$.

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